WEIGHTED COMPOSITION OPERATORS ON WEAK VECTOR-VALUED WEIGHTED BERGMAN SPACES AND HARDY SPACES

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Abstract. In this paper we investigate weighted composition operators between weak and strong vector-valued weighted Bergman spaces and Hardy spaces.

1. **Introduction and Preliminaries**

Weighted composition operators have been studied on different spaces of analytic functions. In [5], Contreras and Hernandez-Diaz have made a study of weighted composition operators on Hardy spaces whereas Mirzakarimi and Siddighi [12] have studied these operators on Bergman and Dirichlet spaces. On Bloch-type spaces, these operators are explored by MacCluer and Zhao [11], Ohno [13], Ohno and Zhao [14] and Ohno, Stroethoff and Zhao [15]. In [8] Kumar studied weighted composition operators between spaces of Dirichlet type by using Carleson measures.

Recently these studies are about spaces of vector-valued analytic functions. For example, in [17], Wang presented some necessary and sufficient conditions for weighted composition operators to be bounded on vector-valued Dirichlet spaces and Laitila, Tylli and Wang [10] studied composition operators from weak to strong vector-valued Bergman spaces Hardy spaces. For some information about vector-valued Bergman spaces see [1, 3].

Let X be a complex Banach space and $\mathbb{D}$ be the open unit ball of $\mathbb{C}$. We consider *weight* as a strictly positive bounded continuous function $v : \mathbb{D} \to \mathbb{R}^+$. Let $p \geq 1$ and v be a weight. The vector-valued weighted Bergman space $A^p_v(X)$ consists of all analytic functions $f : \mathbb{D} \to X$ such that

$$||f||_{A^p_v(X)} = \left(\int_{\mathbb{D}} ||f(z)||^p_X v(z) dA(z) \right)^{\frac{1}{p}} < \infty.$$

where dA is the normalized area measure on $\mathbb{D}$. Also, the vector-valued weighted Hardy space $H^p_v(X)$ consists of all analytic functions $f : \mathbb{D} \to X$ for which

$$||f||_{H^p_v(X)} = \sup_{0<r<1} \left(\int_{\mathbb{T}} ||f(r\zeta)||^p_X v(r\zeta)\ dm(\zeta) \right)^{\frac{1}{p}} < \infty,$$

where $dm(\zeta)$ is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. In the case $X = \mathbb{C}$, we write $A^p_v(X) = A^p_v$ and $H^p_v(X) = H^p_v$. Also, if $v \equiv 1$, then we have $A^p_v(X) = A^p(X)$ and $H^p_v(X) = H^p(X)$. The following weak versions of these spaces were considered by e.g. Blasco [2] and Bonet, Domanski and Lindstrom [4]:

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the weak spaces $wA^p_v(X)$ and $wH^p_v(X)$ consist of all analytic functions $f:\mathbb{D}\to X$ for which

$$||f||_{wA^p_v(X)}=\sup_{||x^*||\leq 1}||x^*\circ f||_{A^p_v},\qquad ||f||_{wH^p_v(X)}=\sup_{||x^*||\leq 1}||x^*\circ f||_{H^p_v},$$

are finite, respectively. Here $x^*\in X^*$, the dual space of X.

Let φ be an analytic self-map of $\mathbb{D}$; that is $\varphi(\mathbb{D})\subset\mathbb{D}$, and u a scaler-valued analytic function on $\mathbb{D}$. We can define the weighted composition operator uC_φ on the space of analytic functions as follows:

$$uC_\varphi(f)(z)=u(z)f(\varphi(z)).$$

When $u(z)\equiv 1$, we just have the composition operator C_φ, defined by $C_\varphi(f)=f\circ\varphi$. Also if $\varphi=I$, the identity function, then we get the multiplication operator M_u defined by $M_u(f)(z)=u(z)f(z)$. It is well known that for every analytic map $\varphi:\mathbb{D}\to\mathbb{D}$, $C_\varphi:A^p(X)\to A^p(X)$ and $C_\varphi:H^p(X)\to H^p(X)$ are bounded, and and also on $wA^p(X), wH^p(X)$. For complete discussion about composition operators we refer to [6, 16]. We consider the infinite dimensional complex Banach space X, since $wA^p(X)=A^p(X)$ and $wH^p(X)=H^p(X)$, for $\alpha>-1$ and any finite dimensional Banach space X.
But for the infinite dimensional complex Banach space X, $A^p(X)\neq wA^p(X)$ ($H^p(X)\neq wH^p(X)$) and $||.||_{wA^p(X)}$ is not equivalent to $||.||_{A^p(X)}$ on $A^p(X)$ ($||.||_{wH^p(X)}$ is not equivalent to $||.||_{H^p(X)}$ on $H^p(X)$), see [10] Proposition 3.1 ([9] Example 15).

Our aim in this paper is to compute the norm of weighted composition operators between $wA^p_v(X)$ and $A^p_v(X)$, for $p\geq 2$ and also between $wH^p_v(X)$ and $H^p_v(X)$, for $p\geq 2$, where v and v' are weights.
Throughout the remainder of this paper, c will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation A $\approx$ B means that there are positive constants c_1 and c_2 such that $c_1A\leq B\leq c_1A$.

2. **Main Results**

Proposition 2.1. Let X be any complex Banach spaces, v be a weight of the form $v=|\mu|$, where μ is an analytic function without any zeros on $\mathbb{D}$, v' be a weight and $1\leq p<\infty$. Then

$$||uC_\varphi:wA^p_v(X)\longrightarrow A^p_{v'}(X)||\leq\left(\int_{\mathbb{D}}\frac{|u(z)|^pv'(z)}{(1-|\varphi(z)|^2)^2v(\varphi(z))}\,dA(z)\right)^{1/p}, and \tag{2.1}$$

(2.2)

$$||uC_\varphi:wH^p_v(X)\longrightarrow H^p_{v'}(X)||\leq\sup_{0<r<1}\left(\int_{\mathbb{T}}\frac{|u(r\zeta)|^pv'(r\zeta)}{(1-|\varphi(r\zeta)|^2)v(\varphi(r\zeta))}\,dm(\zeta)\right)^{1/p}.$$

Proof. By Lemma 2.1 of [18] we have

$$|f(z)|\leq\frac{||f||_{A^p_v}}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}},$$

for any $f\in A^p_v$ and $z\in\mathbb{D}$. Thus, for $f\in wA^p_v(X)$, we have

$$||f(z)||^p_X=\sup_{||x^*||\leq 1}|(x^*\circ f)(z)|^p\leq\frac{1}{(1-|z|^2)^2v(z)}\sup_{||x^*||\leq 1}||x^*\circ f||^p_{A^p_v}$$

$$= \frac{1}{(1-|z|^2)^2 v(z)} ||f||^p_{wA^p_v(X)}.$$

Hence

$$\begin{aligned} ||uC_\varphi f||^p_{A^p_{v'}(X)} &= \int_{\mathbb{D}} |u(z)|^p ||f(\varphi(z))||^p_X v'(z)\, dA(z) \\ &\leq ||f||^p_{wA^p_v(X)} \int_{\mathbb{D}} \frac{|u(z)|^p v'(z)}{(1-|z|^2)^2 v(\varphi(z))}\, dA(z). \end{aligned}$$

The proof of the theorem is complete. □

For the next results we need the following Dvoretzky's well-known theorem.

Lemma 2.2. [7] Suppose that X is an infinite dimensional complex Banach space. Then for any $\epsilon > 0$ and $n \in \mathbb{N}$, there is a linear embedding $T_n : l^2_n \to X$ such that

$$(2.3) \qquad (1+\epsilon)^{-1} \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2} \leq \left\| \sum_{j=1}^n a_j T_n e_j \right\|_X \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2}$$

for any scalars $a_1, a_2, \cdots, a_n$ and some orthonormal basis $\{e_1, \cdots, e_n\}$ of l^2_n.

Now, we prove a lower bound for the operator $uC_\varphi : wA^p_v(X) \longrightarrow A^p_{v'}(X)$, in the case $2 \leq p < \infty$.

Theorem 2.3. Let X be any complex infinite-dimensional Banach space, v be a weight of the form $v = |\mu|$, where μ is an analytic function without any zeros on $\mathbb{D}$, v' be a weight and $2 \leq p < \infty$. Then

$$(2.4) \qquad ||uC_\varphi : wA^p_v(X) \longrightarrow A^p_{v'}(X)|| \approx \left(\int_{\mathbb{D}} \frac{|u(z)|^p v'(z)}{(1-|\varphi(z)|^2)^2 v(\varphi(z))}\, dA(z) \right)^{1/p}.$$

Proof. We only prove there exists a positive constant c such that

$$||uC_\varphi : wA^p_v(X) \longrightarrow A^p_{v'}(X)|| \geq c \left(\int_{\mathbb{D}} \frac{|u(z)|^p v'(z)}{(1-|\varphi(z)|^2)^2 v(\varphi(z))}\, dA(z) \right)^{1/p}.$$

Suppose that $x \in X$ with $||x|| = 1$ and define $g : \mathbb{D} \to X$ by $g(z) = \frac{1}{\mu(z)^{\frac{1}{p}}} x$. Then g is an analytic function on $\mathbb{D}$, and $||g||_{wA^p_v(X)} = 1$, so that

$$||uC_\varphi||^p \geq ||ug \circ \varphi||^p_{A^p_{v'}} = \int_{\mathbb{D}} \frac{|u(z)|^p v'(z)}{v(\varphi(z))}\, dA(z).$$

Hence

$$\int_{\{z \in \mathbb{D}: |\varphi(z)|^2 < 1/2\}} \frac{|u(z)|^p v'(z)}{(1-|\varphi(z)|^2)^2 v(\varphi(z))}\, dA(z) \leq 4 \int_{\mathbb{D}} \frac{|u(z)|^p v'(z)}{v(\varphi(z))}\, dA(z) \leq 4 ||uC_\varphi||^p.$$

So, it will be sufficient to prove that there exists a positive constant c such that

$$||uC_\varphi||^p \geq c \int_{\{z \in \mathbb{D}: |\varphi(z)|^2 \geq 1/2\}} \frac{|u(z)|^p v'(z)}{(1-|\varphi(z)|^2)^2 v(\varphi(z))}\, dA(z).$$

Let $\lambda_k = k^{2/p - 1/2}$, for any $n \in \mathbb{N}$, we define functions f_n as follows

$$f_n(z) = \frac{1}{\mu(z)^{\frac{1}{p}}} \sum_{k=1}^n \lambda_k z^k T_n e_k,$$

where the linear embedding T_n is the same as in Lemma 2.2, $||T_n|| = 1$ and $||T_n^{-1}|| \leq (1+\epsilon)$ and $(e_1, \cdots, e_n)$ is an orthonormal basis of ℓ_2^n. As in the proof of Theorem 3.2 [10], there exists $c > 0$ such that for X^* with $||x^*|| \leq 1$, we have

$$\begin{aligned}\|x^* \circ f_n\|_{A_v^p} =& \| \frac{1}{\mu(z)^{\frac{1}{p}}} \sum_{k=1}^{n} \lambda_k z^k x^* T_n e_k \|_{A_v^p} \\ =& \| \sum_{k=1}^{n} \lambda_k x^*(T_n e_k) z^k \|_{A^p} \\ \leq& c \left(\sum_{k=1}^{n} |x^*(T_n e_k)|^2 \right)^{1/2} \leq c.\end{aligned}$$

It follows that $\|f_n\|_{wA_v^p(X)} \leq c$. Thus, Fatou's lemma implies that

$$\begin{aligned}\|uC_\varphi\|^p \geq& c^{-p} \limsup_{n\to\infty} \|uC_\varphi f_n\|^p_{A^p_{v'}(X)} \\ =& c^{-p} \limsup_{n\to\infty} \int_{\mathbb{D}} |u(z)|^p \| \frac{1}{\mu(\varphi(z))^{\frac{1}{p}}} \sum_{k=1}^{n} \lambda_k \varphi(z)^k T_n e_k \|_X^p v'(z)\, dA(z) \\ =& c^{-p} \limsup_{n\to\infty} \int_{\mathbb{D}} \| \sum_{k=1}^{n} \lambda_k \varphi(z)^k T_n e_k \|_X^p \frac{|u(z)|^p v'(z)}{v(\varphi(z))}\, dA(z) \\ \geq& \frac{c^{-p}}{(1+\epsilon)^p} \limsup_{n\to\infty} \int_{\mathbb{D}} \left(\sum_{k=1}^{n} k^{4/p-1} |\varphi(z)|^{2k} \right)^{p/2} \frac{|u(z)|^p v'(z)}{v(\varphi(z))}\, dA(z) \\ =& \frac{c^{-p}}{(1+\epsilon)^p} \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} k^{4/p-1} |\varphi(z)|^{2k} \right)^{p/2} \frac{|u(z)|^p v'(z)}{v(\varphi(z))}\, dA(z) \\ \geq& \frac{c_1 c^{-p}}{(1+\epsilon)^p} \int_{\{z\in\mathbb{D}: |\varphi(z)|^2 \geq 1/2\}} \frac{|u(z)|^p v'(z)}{(1-|\varphi(z)|^2)^2 v(\varphi(z))}\, dA(z)\end{aligned}$$

and the last inequality is derived by Lemma 2.3 [10]. As $\epsilon > 0$ was arbitrary, we obtain the desired lower bound estimate.
□

Theorem 2.4. Let X be any complex infinite-dimensional Banach space, v be a weight of the form $v = |\mu|$, where μ is an analytic function without any zeros on $\mathbb{D}$, v' be a weight and $2 \leq p < \infty$. Then

$$||uC_\varphi : wH_v^p(X) \longrightarrow H_{v'}^p(X)|| \approx \left(\int_{\mathbb{T}} \frac{|u(\zeta)|^p v'(\zeta)}{(1-|\varphi(\zeta)|^2) v(\varphi(\zeta))}\, dm(\zeta) \right)^{1/p}. \tag{2.5}$$

Proof. Similar to the proof of previous theorem, we only prove that there exists $c > 0$ such that

$$||uC_\varphi||^p \geq c \int_{\{\zeta\in\mathbb{T}: |\varphi(r\zeta)|^2 \geq 1/2\}} \frac{|u(r\zeta)|^p v'(r\zeta)}{(1-|\varphi(r\zeta)|^2) v(\varphi(r\zeta))}\, dm(\zeta).$$

Let $\lambda_k = k^{1/p-1/2}$ and define

$$f_n(z) := \frac{1}{\mu(z)^{\frac{1}{p}}} \sum_{k=1}^{n} \lambda_k z^k T_n e_k,$$

where the linear embedding T_n is the same as in Lemma 2.2, $||T_n|| = 1$ and $||T_n^{-1}|| \leq (1+\epsilon)$ and $(e_1, \cdots, e_n)$ is an orthonormal basis of ℓ_2^n. As in the proof of Theorem 2.2 [10], there exists $c > 0$ such that for X^* with $||x^*|| \leq 1$, we have

$$\begin{aligned}\|x^* \circ f_n\|_{H_v^p} =&\| \frac{1}{\mu(z)^{\frac{1}{p}}} \sum_{k=1}^{n} \lambda_k z^k x^* T_n e_k \|_{H_v^p} \\ =&\| \sum_{k=1}^{n} \lambda_k x^*(T_n e_k) z^k \|_{H^p} \\ \leq& c \left(\sum_{k=1}^{n} |x^*(T_n e_k)|^2 \right)^{1/2} \leq c.\end{aligned}$$

Thus $\|f_n\|_{wH_v^p(X)} \leq c$ and by suing Fatou's lemma and Lemma 2.3 [10], we have

$$\begin{aligned}\|uC_\varphi\|^p \geq& c^{-p} \limsup_{n\to\infty} \|uC_\varphi f_n\|^p_{H^p_{v'}(X)} \\ =& c^{-p} \limsup_{n\to\infty} \int_{\mathbb{T}} |u(r\zeta)|^p \| \frac{1}{\mu(\varphi(r\zeta))^{\frac{1}{p}}} \sum_{k=1}^{n} \lambda_k \varphi(r\zeta)^k T_n e_k \|_X^p v'(r\zeta)\, dm(\zeta) \\ =& c^{-p} \limsup_{n\to\infty} \int_{\mathbb{T}} \| \sum_{k=1}^{n} \lambda_k \varphi(r\zeta)^k T_n e_k \|_X^p \frac{|u(r\zeta)|^p v'(r\zeta)}{v(\varphi(r\zeta))}\, dm(\zeta) \\ \geq& \frac{c^{-p}}{(1+\epsilon)^p} \limsup_{n\to\infty} \int_{\mathbb{T}} \left(\sum_{k=1}^{n} k^{2/p-1} |\varphi(r\zeta)|^{2k} \right)^{p/2} \frac{|u(r\zeta)|^p v'(r\zeta)}{v(\varphi(r\zeta))}\, dm(\zeta) \\ =& \frac{c^{-p}}{(1+\epsilon)^p} \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} k^{2/p-1} |\varphi(r\zeta)|^{2k} \right)^{p/2} \frac{|u(r\zeta)|^p v'(r\zeta)}{v(\varphi(r\zeta))}\, dm(\zeta) \\ \geq& \frac{c_1 c^{-p}}{(1+\epsilon)^p} \int_{\{z\in\mathbb{T}: |\varphi(r\zeta)|^2 \geq 1/2\}} \frac{|u(r\zeta)|^p v'(r\zeta)}{(1-|\varphi(r\zeta)|^2) v(\varphi(r\zeta))}\, dm(\zeta).\end{aligned}$$

Take $\epsilon = 1$, then

$$\|uC_\varphi\|^p \geq c \int_{\mathbb{T}} \frac{|u(r\zeta)|^p v'(r\zeta)}{(1-|\varphi(r\zeta)|^2) v(\varphi(r\zeta))}\, dm(\zeta).$$

As $r \to 1$,

$$\begin{aligned}\|uC_\varphi\|^p \geq& c \limsup_{r\to 1} \int_{\mathbb{T}} \frac{|u(r\zeta)|^p v'(r\zeta)}{(1-|\varphi(r\zeta)|^2) v(\varphi(r\zeta))}\, dm(\zeta) \\ \geq& c \int_{\mathbb{T}} \frac{|u(\zeta)|^p v'(\zeta)}{(1-|\varphi(\zeta)|^2) v(\varphi(\zeta))}\, dm(\zeta).\end{aligned}$$

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